On estimates of solutions for Helmholtz equations
and their applications *

Hideo Nakazawa
Department of Mathematics, Nippon Medical School

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1 Problems and results

Assume that \( N \geq 2 \) and consider the following Helmholtz equations in an exterior domain with star-shaped boundary:

\[
\begin{aligned}
(−\Delta − \kappa^2)u &= f(x), & x &\in \Omega, \\
\quad u &= 0, & x &\in \partial\Omega,
\end{aligned}
\]  

(1.1)

where \( u = u(x, \kappa), \ k = \sigma + ir \in \mathbb{C} \) and \( \mathbb{R}^N \setminus \Omega \) is star-shaped with respect to the origin \( 0 \not\in \Omega \) (\( \subset \mathbb{R}^N \)), i.e., \( \left( \frac{x}{r}, n \right) \leq 0 \) for unit outer normal \( n \) of \( \partial\Omega \). In the following, we exclude the case \( \Omega = \mathbb{R}^N \) and assume that \( \min\{|x|; x \in \partial\Omega\} > r_0 \) for some \( r_0 > 0 \). The method described here can be adapted also to stationary Schrödinger equations with magnetic fields or stationary dissipative wave equations.

Our aim is to establish the uniform resolvent estimates for (1.1) in \( \Omega \). The main task is to obtain the positivity of the energy form obtained from Laplacian, which was violated for the case \( N = 2 \) in the existing research (e.g., Ikebe-Saito [2], Mochizuki [6]) due to the term \( \frac{(N-1)(N-3)}{4r^2}|u|^2 \) (\( N \) denotes the space dimension and \( r = |x| \)).

The first key for conquering this difficulty was the Hardy type inequalities related to radiation conditions:

\[
\left| \left| D^\pm u \right| \right| < \infty \quad \text{where} \quad D^\pm u = D^\pm u \cdot \frac{x}{r}, \quad D^\pm u = \nabla u + \left( \frac{N-1}{2r} u \mp iku \right) \frac{x}{r} \quad (\Im \kappa \geq 0).
\]

Proposition 1.1 ([7, 9]) Assume that \( \Omega \subseteq \mathbb{R}^N \) is a general domain with smooth boundary and \( N \geq 1 \). For \( v \in C_0^\infty(\Omega) \), the following inequalities hold:

\[
\left| \left| v_r + \frac{N-1}{2r} v \right| \right|_\phi^2 \geq \left| \left| \phi \right| \right|_{h_\alpha}^2, \quad \left| \left| D^\pm_r v \right| \right|_\phi^2 \geq \pm \Im \kappa \left| \left| v \right| \right|_{h_\alpha}^2 + \left| \left| v \right| \right|_{h_\alpha}^2, \quad (\Im \kappa \geq 0)
\]

where

\[
h_\alpha(r) = -\frac{a\phi'(r)}{r} - \frac{a(a-1)\phi(r)}{r^2},
\]

and \( \left| \left| v \right| \right|_w^2 = \int_\Omega w(x)|v(x)|^2 dx \) denotes the weighted \( L^2 \)-norm with some weight function \( w = w(x) \geq 0 \).

If these are used, the positivity in the case of \( N = 2 \) is re-established. However, the duality on the weight function is violated:

Theorem 1.2 ([9]) For a solution \( u \) of (1.1), the following uniform resolvent estimate holds:

\[
|\kappa| \left| \left| u \right| \right|_{r^{-1-s}}^2 + \Im \kappa \left| \left| u \right| \right|_{r^{-1}}^2 + \left| \left| u \right| \right|_{r^{-3-s}}^2 + \left| \left| D^\pm_r u \right| \right|^2 \leq C_3 \left| \left| f \right| \right|_{r^{3+s}}^2, \quad (\Im \kappa \geq 0)
\]

\(^*\)This is based on joint work with Kiyoshi Mochizuki (Emeritus, Tokyo Metropolitan University).

\(^1\)The radiation condition is a boundary condition at infinity to ensure the uniqueness of the solution of (1.1).

\(^2\)The similar result also holds for stationary dissipative wave and Schrödinger equations under suitable conditions.
Note that (1.2) is a global resolvent estimate with the spectral parameter \( \kappa \). In this sense, it differs from the local (low or high energy) inequality derived by e.g., Ikebe-Saito [2], Agmon [1] or Kuroda [4], etc. The global resolvent estimate has already been proved by Mochizuki [6]. On the other hand, the uniform estimate is established by Kato-Yajima [3] for Helmholtz equations in the whole space. It is extended by Mochizuki [7] to magnetic Schrödinger equations in an exterior domain in \( \mathbb{R}^N \) with \( N \geq 3 \).

One key for the duality is a form of the weight function which appears in the Hardy type inequality, in a two-dimensional exterior domain, which originally proved by J. Leray [5]:

**Proposition 1.3** ([5]) For any \( \psi \in C_0^\infty (\Omega) \), it holds that

\[
||\psi||^2 \left( 2r \log \frac{r_0}{r} \right)^{-2} \leq C ||\psi||^2 \left( \int_\Omega \frac{||\psi(x)||^2}{4r^2 \left( \log \frac{r}{r_0} \right)^2} dx \right) \leq ||\psi||^2.
\]

Our results are given in the following form:

**Theorem 1.4** ([8]) For a solution \( u \) of (1.1) and for each \( \kappa \in \Pi_+ = \{ \kappa \in \mathbb{C}; \pm \Re \kappa > 0, \Im \kappa > 0 \} \), the following inequality holds:

\[
||u||^2 \left( \frac{1 + \log \frac{r}{r_0}}{r} \right)^{-2} + 3\kappa ||u||^2 \left( 1 + \log \frac{r}{r_0} \right)^{-2} \leq C_1 ||f||^2 \left( \frac{1}{r^2 \left( \frac{1 + \log \frac{r}{r_0}}{r} \right)^{2}} \right).
\]

Moreover, for vector-valued function \( D^+ u(x) \) and for each \( \kappa \in \Pi_+ \), the following inequality holds:

\[
||D^+ u||^2 \left( \frac{1}{4 + \log \frac{r}{r_0}} \right)^{-2} + 3 \kappa ||D^+ u||^2 \left( 4 + \log \frac{r}{r_0} \right)^{-2} \leq C_2 ||f||^2 \left( \frac{1}{r^2 \left( 4 + \log \frac{r}{r_0} \right)^{2}} \right).
\]

**Theorem 1.5** ([8]) Assume that the function \( \mu(r) \) is smooth, non-negative and integrable on \([r_0, \infty)\) satisfying

\[
2r \mu'(r) \leq \mu(r) \quad \text{and} \quad \mu(r) \leq \frac{C_3}{\left( 4 + \log \frac{r}{r_0} \right)^2}
\]

for some \( C_3 > 0 \). Then for a solution \( u \) of (1.1) and for each \( \kappa \in \Pi_+ \), the following inequality holds:

\[
|\kappa|^2 ||u||^2 + ||\nabla u||^2 \mu \leq C_4 ||f||^2 \left( \frac{1}{r^2 \left( 1 + \log \frac{r}{r_0} \right)^{2}} + \mu(r)^{-1} \right).
\]

**References**


